Fredholm Property of Nonlocal Problems for Integro-Differential Hyperbolic Systems

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Abstract

The paper concerns nonlocal time-periodic boundary value problems for first-order integro-differential hyperbolic systems with boundary inputs. The systems are subjected to integral boundary conditions. Under natural regularity assumptions on the data, it is shown that the problems display completely non-resonant behavior and satisfy the Fredholm alternative in the spaces of continuous and time-periodic functions.

Key words: first-order integro-differential hyperbolic systems, integral boundary conditions, time-periodic solutions, Fredholm alternative, non-resonant behavior.

Mathematics Subject Classification: 35A17, 35B10, 35F45, 35L40

1   Introduction

1.1   Motivation

We consider the following first-order integro-differential hyperbolic system in one space variable

$$\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x,t)u_k + \sum_{k=1}^{n}\int_0^x g_{jk}(y,t)u_k(y,t)dy$$

$$= \sum_{k=m+1}^{n} h_{jk}(x,t)u_k(0,t) + \sum_{k=1}^{m} h_{jk}(x,t)u_k(1,t) + f_j(x,t), \quad x \in (0,1),\ j \le n,$$

$$(1.1)$$

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subjected to periodic conditions in time

$$u_j(x,t) = u_j(x, t + 2\pi), \quad j \le n, \tag{1.2}$$

and integral boundary conditions in space

$$
\begin{aligned}
u_j(0,t) &= \sum_{k=1}^{n} \int_0^1 r_{jk}(x,t) u_k(x,t)\, dx, \quad 1 \le j \le m, \\
u_j(1,t) &= \sum_{k=1}^{n} \int_0^1 r_{jk}(x,t) u_k(x,t)\, dx, \quad m < j \le n,
\end{aligned}
\tag{1.3}
$$

where $0 \le m \le n$ are positive integers.

Note that the boundary terms $u_{m+1}(0,t), \dots, u_n(0,t)$ and $u_1(1,t), \dots, u_m(1,t)$ contribute into the differential system (1.1), while the boundary terms $u_1(0,t), \dots, u_m(0,t)$ and $u_{m+1}(1,t), \dots, u_n(1,t)$ contribute into the boundary conditions (1.3). In this form, which is motivated by applications, the problem has been studied in [13, 18].

The Volterra integral terms in (1.1) are motivated by the aforementioned applications (see, e.g., [13, 18]). As it will be seen from our proof of Theorem 1.2, our analysis applies also to the case when these terms are replaced by the Fredholm integral terms.

In general, systems of the type (1.1), (1.3) model a broad range of physical problems such as traffic flows, chemical reactors and heat exchangers [18]. They are also used to describe problems of population dynamics (see, e.g., [3, 7, 15, 20] and references therein) and polymer rheology [4]. Moreover, they appear in the study of optimal boundary control problems [13, 16, 18, 19].

Establishing a Fredholm property is a first step in developing a theory of local smooth continuation [12] and bifurcation [1, 2, 11] for Fredholm hyperbolic operators, in particular, such tools as Lyapunov-Schmidt reduction. Buono and Eftimie [1] consider autonomous 2×2 nonlocal hyperbolic systems in a single space variable, describing formation and movement of various animal, cell and bacterial aggregations, with some biologically motivated integral terms in the differential equations. One of the main results in [1] is a Fredholm alternative for the linearizations at a steady-state, which enables performing a bifurcation analysis by means of the Lyapunov-Schmidt reduction. Here we continue this line of research, establishing the Fredholm property for a wide range of non-autonomous nonlocal problems for $(n \times n)$-hyperbolic systems, with nonlocalities both in the differential equations and in the boundary conditions.

We show that the problem (1.1)–(1.3) demonstrates a completely non-resonant behavior (in other terms, no *small divisors* occur). More precisely, we prove the Fredholm alternative for (1.1)–(1.3) under the only assumptions that the coefficients in (1.1) and (1.3) are sufficiently smooth and a kind of Levy condition is fulfilled. The proof extends the ideas of [9, 10] for proving the Fredholm alternative for first-order one-dimensional hyperbolic systems with reflection boundary conditions, and also the ideas of [8] for proving a smoothing property for boundary value hyperbolic problems. In contrast to [9] and [10], where conditions excluding a resonant behavior are imposed, the present Fredholmness result is unconditional, in this respect.

1.2 Our result

By $C_{n,2\pi}$ we denote the vector space of all 2π-periodic in t and continuous maps $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$, with the norm

$$\|u\|_\infty = \max_{j \le n} \max_{x \in [0,1]} \max_{t \in \mathbb{R}} |u_j|.$$

Similarly, $C^1_{n,2\pi}$ denotes the Banach space of all $u \in C_{n,2\pi}$ such that $\partial_x u, \partial_t u \in C_{n,2\pi}$, with the norm

$$\|u\|_1 = \|u\|_\infty + \|\partial_x u\|_\infty + \|\partial_t u\|_\infty.$$

For simplicity, we skip subscript n if $n = 1$ and write $C_{2\pi}$ and $C^1_{2\pi}$ for $C_{1,2\pi}$ and $C^1_{1,2\pi}$, respectively.

We make the following natural assumptions on the coefficients of (1.1) and (1.3):

$$a_j \in C^1_{2\pi} \text{ and } b_{jk}, \partial_t b_{jk}, g_{jk}, h_{jk}, r_{jk}, \partial_t r_{jk} \in C_{2\pi} \text{ for all } j \le n \text{ and } k \le n, \qquad (1.4)$$

$$a_j \ne 0 \text{ for all } (x,t) \in [0,1] \times \mathbb{R} \text{ and } j \le n, \qquad (1.5)$$

and

$$\begin{aligned} &\text{for all } 1 \le j \ne k \le n \text{ there exists } \tilde{b}_{jk} \in C_{2\pi} \text{ such that} \\ &\partial_t \tilde{b}_{jk} \in C_{2\pi} \text{ and } b_{jk} = \tilde{b}_{jk}(a_k - a_j). \end{aligned} \qquad (1.6)$$

The assumption (1.5) is standard and means the non-degeneracy of the hyperbolic system (1.1). The assumption (1.6) is a kind of the well-known Levy condition appearing in various aspects of the hyperbolic theory, for instance, for proving the spectrum-determined growth condition for semiflows generated by initial value problems for hyperbolic systems [5, 14, 17]. It plays also a crucial role in the Fredholm analysis of hyperbolic PDEs (see Example 1.3 below).

Given $j \le n$, $x \in [0,1]$, and $t \in \mathbb{R}$, the j-th characteristic of (1.1) is defined as the solution $\xi \in [0,1] \mapsto \omega_j(\xi, x, t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \omega_j(\xi, x, t) = \frac{1}{a_j(\xi, \omega_j(\xi, x, t))}, \quad \omega_j(x, x, t) = t. \qquad (1.7)$$

To shorten notation, we will write $\omega_j(\xi) = \omega_j(\xi, x, t)$. In what follows we will use the equalities

$$\partial_x \omega_j(\xi) = -\frac{1}{a_j(x,t)} \exp \int_\xi^x \left(\frac{\partial_2 a_j}{a_j^2}\right)(\eta, \omega_j(\eta)) \, d\eta \qquad (1.8)$$

and

$$\partial_t \omega_j(\xi) = \exp \int_\xi^x \left(\frac{\partial_2 a_j}{a_j^2}\right)(\eta, \omega_j(\eta)) \, d\eta, \qquad (1.9)$$

where by ∂_i here and below we denote the partial derivative with respect to the i-th argument. Set

$$\begin{aligned} c_j(\xi, x, t) &= \exp \int_x^\xi \left(\frac{b_{jj}}{a_j}\right)(\eta, \omega_j(\eta)) \, d\eta, \\ d_j(\xi, x, t) &= \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi))}, \end{aligned} \qquad (1.10)$$

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and

$$x_j = \begin{cases} 0 & \text{if } 1 \le j \le m, \\ 1 & \text{if } m < j \le n. \end{cases}$$

Integration along the characteristic curves brings the system (1.1)–(1.3) to the integral form

$$
\begin{aligned}
u_j(x,t) \;=\; & c_j(x_j, x, t) \sum_{k=1}^{n} \int_0^1 r_{jk}(\eta, \omega_j(x_j)) u_k(\eta, \omega_j(x_j)) \, d\eta \\
& - \sum_{k \ne j} \int_{x_j}^{x} d_j(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) \, d\xi \\
& - \sum_{k=1}^{n} \int_{x_j}^{x} d_j(\xi, x, t) \int_0^\xi g_{jk}(y, \omega_j(\xi)) u_k(y, \omega_j(\xi)) \, dy d\xi \qquad (1.11) \\
& + \sum_{k=1}^{n} \int_{x_j}^{x} d_j(\xi, x, t) h_{jk}(\xi, \omega_j(\xi)) u_k(1 - x_k, \omega_j(\xi)) \, d\xi \\
& + \int_{x_j}^{x} d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) \, d\xi, \quad j \le n.
\end{aligned}
$$

Indeed, let u be a C^1-solution to (1.1)–(1.3). Then, using (1.1) and (1.7), for all $j \le n$ we have

$$
\frac{d}{d\xi} u_j(\xi, \omega_j(\xi)) = \partial_1 u_j(\xi, \omega_j(\xi)) + \frac{\partial_2 u_j(\xi, \omega_j(\xi))}{a_j(\xi, \omega_j(\xi))}
$$

$$
= \frac{1}{a_j(\xi, \omega_j(\xi))} \left(-\sum_{k=1}^{n} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) + \sum_{k=m+1}^{n} h_{jk}(\xi, \omega_j(\xi)) u_k(0, \omega_j(\xi)) \right.
$$

$$
\left. + \sum_{k=1}^{m} h_{jk}(\xi, \omega_j(\xi)) u_k(1, \omega_j(\xi)) - \sum_{k=1}^{n} \int_0^\xi g_{jk}(y, \omega_j(\xi)) u_k(y, \omega_j(\xi)) \, dy + f_j(\xi, \omega_j(\xi)) \right).
$$

This is a linear inhomogeneous ordinary differential equation for the function $u_j(\cdot, \omega_j(\cdot, x, t))$, and the variation of constants formula (with initial condition at x_j) gives

$$
u_j(x,t) = u_j(x_j, \omega_j(x_j)) \exp \int_x^{x_j} \left(\frac{b_{jj}}{a_j} \right) (\xi, \omega_j(\xi)) \, d\xi - \int_x^{x_j} \exp \int_\xi^x \left(\frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta)) \, d\eta
$$

$$
\times \frac{1}{a_j(\xi, \omega_j(\xi))} \left(-\sum_{k \ne j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) + \sum_{k=m+1}^{n} h_{jk}(\xi, \omega_j(\xi)) u_k(0, \omega_j(\xi)) \right.
$$

$$
\left. + \sum_{k=1}^{m} h_{jk}(\xi, \omega_j(\xi)) u_k(1, \omega_j(\xi)) - \sum_{k=1}^{n} \int_0^\xi g_{jk}(y, \omega_j(\xi)) u_k(y, \omega_j(\xi)) \, dy + f_j(\xi, \omega_j(\xi)) \right) d\xi.
$$

Inserting the boundary conditions (1.3) and using the notation (1.10), we get (1.11), as desired.

Definition 1.1 A function $u \in C_{n,2\pi}$ is called a *continuous solution* to (1.1)–(1.3) if it satisfies (1.11).

Our result states that either the space of nontrivial solutions to (1.1)–(1.3) with $f = (f_1, ..., f_n) = 0$ is not empty and has finite dimension or the system (1.1)–(1.3) has a unique solution for any f.

Theorem 1.2 *Suppose that the conditions (1.4)–(1.6) are fulfilled. Let $\mathcal{K}$ denote the vector space of all continuous solutions to (1.1)–(1.3) with $f \equiv 0$. Then*
   (i) $\dim \mathcal{K} < \infty$ and the vector space of all $f \in C_{n,2\pi}$ such that there exists a continuous solution to (1.1)–(1.3) is a closed subspace of codimension $\dim \mathcal{K}$ in $C_{n,2\pi}$.
   (ii) If $\dim \mathcal{K} = 0$, then for any $f \in C_{n,2\pi}$ there exists a unique continuous solution u to (1.1)–(1.3).

Example 1.3 Consider the following example showing that the condition (1.6) plays a crucial role for our result:

$$\partial_t u_1 + \frac{2}{\pi}\partial_x u_1 - u_2 = 0$$
$$\partial_t u_2 + \frac{2}{\pi}\partial_x u_2 + u_1 = 0,$$
(1.12)

$$u_1(x,t) = u_1(x, t+2\pi), \quad u_2(x,t) = u_2(x, t+2\pi),$$
(1.13)

$$u_1(0,t) = 0, \quad u_2(1,t) = 0.$$
(1.14)

This problem is a particular case of (1.1)–(1.3) and satisfies all assumptions of Theorem 1.2 with the exception of (1.6). It is straightforward to check that

$$u_1 = \sin\frac{\pi}{2}x \sin l\left(t - \frac{\pi}{2}x\right), \quad u_2 = \cos\frac{\pi}{2}x \sin l\left(t - \frac{\pi}{2}x\right), \quad l \in \mathbb{N},$$

are infinitely many linearly independent solutions to the problem (1.12)–(1.14) and, therefore, the kernel of the operator of (1.12)–(1.14) is infinite dimensional. Thus, the conclusion of Theorem 1.2 is not true without (1.6).

2 Proof of Theorem 1.2

Define linear bounded operators $R, B, G, H, F : C_{n,2\pi} \to C_{n,2\pi}$ by

$$(Ru)_j(x,t) = c_j(x_j, x, t)\sum_{k=1}^{n}\int_0^1 r_{jk}(\eta, \omega_j(x_j))u_k(\eta, \omega_j(x_j))\,d\eta, \quad j \le n,$$

$$(Bu)_j(x,t) = -\sum_{k \ne j}\int_{x_j}^x d_j(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))u_k(\xi, \omega_j(\xi))\,d\xi, \quad j \le n,$$
(2.1)

$$(Gu)_j(x,t) = -\sum_{k=1}^{n}\int_{x_j}^x\int_0^\xi d_j(\xi, x, t)g_{jk}(y, \omega_j(\xi))u_k(y, \omega_j(\xi))\,dyd\xi, \quad j \le n,$$
(2.2)

$$(Hu)_j(x,t) = \sum_{k=1}^{n}\int_{x_j}^x d_j(\xi, x, t)h_{jk}(\xi, \omega_j(\xi))u_k(1 - x_k, \omega_j(\xi))\,d\xi, \quad j \le n,$$
(2.3)

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and

$$(Ff)_j(x,t) = \int_{x_j}^{x} d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) \, d\xi, \quad j \leq n.$$

Then the system (1.11) can be written as the operator equation

$$u = Ru + Bu + Gu + Hu + Fu.$$

Note that Theorem 1.2 says exactly that the operator $I - R - B - G - H : C_{n,2\pi} \to C_{n,2\pi}$ is Fredholm of index zero. Nikolsky's criterion [6, Theorem XIII.5.2] says that an operator $I + K$ on a Banach space is Fredholm of index zero whenever K^2 is compact. It is interesting to note that the compactness of K^2 and the identity $I - K^2 = (I + K)(I - K)$ imply that the operator $I - K$ is a parametrix of the operator $I + K$ (see [21]).

We, therefore, have to show that the operator $K^2 : C_{n,2\pi} \to C_{n,2\pi}$ for $K^2 = (R + B + G + H)^2$ is compact. Since the operators $R, B, G,$ and H are bounded and the composition of a bounded and a compact operator is compact, it is enough to show that

$$\text{the operators } \ H, G, R^2, RB, B^2, BR : C_{n,2\pi} \to C_{n,2\pi} \ \text{ are compact.} \qquad (2.4)$$

We start with the compactness of H. By $C_{2\pi}(\mathbb{R})$ we denote the space of all continuous and 2π-time-periodic maps $v : \mathbb{R} \to \mathbb{R}$. Fix arbitrary $j \leq n$ and $k \leq n$ and define the operator $H_{jk} \in \mathcal{L}(C_{2\pi}(\mathbb{R}), C_{2\pi})$ by

$$(H_{jk}v)(x,t) = \int_{x_j}^{x} d_j(\xi, x, t) h_{jk}(\xi, \omega_j(\xi)) v(\omega_j(\xi)) \, d\xi. \qquad (2.5)$$

It suffices to show the compactness of H_{jk}. Change the variable ξ to $z = \omega_j(\xi)$ and denote the inverse map by $\xi = \tilde{\omega}_j(z) = \tilde{\omega}_j(z, x, t)$. Afterwards (2.5) reads

$$(H_{jk}v)(x,t) = \int_{\omega_j(x_j)}^{t} d_j(\tilde{\omega}_j(z), x, t) h_{jk}(\tilde{\omega}_j(z), z) a_j(\tilde{\omega}_j(z), z) v(z) \, dz. \qquad (2.6)$$

By the regularity assumption (1.4), the functions $\omega_j(x_j)$, $\tilde{\omega}_j(z)$, $d_j(\xi, x, t)$, $h_{jk}(x, z)$, and $a_j(x, z)$ are continuous in all their arguments and 2π-periodic in t and, hence, are uniformly continuous in x and t. Then the equicontinuity property of $(H_{jk}v)(x, t)$ for v over a bounded subset of $C_{2\pi}(\mathbb{R})$ straightforwardly follows. Using the Arzela-Ascoli precompactness criterion, we conclude that H_{jk} and, hence, H are compact.

Now we consider the operator G. Changing the variable ξ to $z = \omega_j(\xi, x, t)$ in (2.2), we get

$$(Gu)_j(x,t) = -\sum_{k=1}^{n} \int_{\omega_j(x_j)}^{t} \int_{0}^{\tilde{\omega}_j(z)} d_j(\tilde{\omega}_j(z), x, t) g_{jk}(y, z) a_j(\tilde{\omega}_j(z), z) u_k(y, z) \, dy dz. \qquad (2.7)$$

Similarly to the above, the functions $\omega_j(x_j), \tilde{\omega}_j(z), d_j(\tilde{\omega}_j(z), x, t),$ and $a_j(\tilde{\omega}_j(z), z)$ are 2π-periodic in t and uniformly continuous in x and t. This entails the equicontinuity property for $(Gu)_j(x, t)$ for u over a bounded subset of $C_{n,2\pi}$. The compactness of G again follows from the Arzela-Ascoli theorem.

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We further proceed with the compactness of R^2. For $j \leq n$ and $k \leq n$ define operators $R_{jk} \in \mathcal{L}(C_{2\pi})$ by

$$(R_{jk}w)(x,t) = c_j(x_j, x, t) \int_0^1 r_{jk}(\eta, \omega_j(x_j))w(\eta, \omega_j(x_j))\, d\eta.$$

Fix arbitrary $j \leq n$, $k \leq n$, and $i \leq n$. We prove the compactness of the operator $R_{jk}R_{ki}$; the compactness of all other operators contributing into the R^2 will follow from the same argument. Introduce operators $P_j, Q_{jk} : C_{2\pi} \to C_{2\pi}$ by

$$(P_j w)(x,t) = c_j(x_j, x, t) \int_0^1 w(\eta, t)\, d\eta, \tag{2.8}$$

$$(Q_{jk} w)(x,t) = r_{jk}(x, \omega_j(x_j))w(x, \omega_j(x_j)). \tag{2.9}$$

Then we have

$$R_{jk} = P_j Q_{jk}, \quad R_{ki} = P_k Q_{ki}$$

and, hence

$$R_{jk}R_{ki} = P_j Q_{jk} P_k Q_{ki}.$$

We aim at showing the compactness of $P_j Q_{jk} P_k$, as this and the boundedness of Q_{ki} will entail the compactness of $R_{jk}R_{ki}$. The operator $P_j Q_{jk} P_k$ reads

$$(P_j Q_{jk} P_k w)(x,t) = c_j(x_j, x, t)$$
$$\times \int_0^1 r_{jk}(\xi, \omega_j(x_j, \xi, t))c_k(x_k, \xi, \omega_j(x_j, \xi, t)) \int_0^1 w(\eta, \omega_k(x_k, \xi, t))\, d\eta d\xi. \tag{2.10}$$

Changing the variable ξ to $z = \omega_k(x_k, \xi, t)$, we get

$$(P_j Q_{jk} P_k w)(x,t) = c_j(x_j, x, t)$$
$$\times \int_{\omega_k(x_k,0,t)}^{\omega_k(x_k,1,t)} r_{jk}(\tilde{\omega}_k(t, x_k, z), z)c_k(x_k, \tilde{\omega}_k(t, x_k, z), z) \int_0^1 \partial_3 \tilde{\omega}_k(t, x_k, z)w(\eta, z)\, d\eta dz, \tag{2.11}$$

where

$$\partial_3 \tilde{\omega}_k(\tau, x, t) = a_k(x, t) \exp \int_\tau^t \partial_1 a_k(\tilde{\omega}_k(\rho, x, t), \rho)\, d\rho. \tag{2.12}$$

Similarly to the above, the compactness of $P_j Q_{jk} P_k$ now immediately follows from the regularity assumption (1.4) and the Arzela-Ascoli theorem.

Now we treat the operator

$$(RBu)_j(x,t) = -c_j(x_j, x, t) \sum_{k \neq l} \int_0^1 \int_{x_k}^\eta r_{jk}(\eta, \omega_j(x_j))d_k(\xi, \eta, \omega_j(x_j))$$

$$\times b_{kl}(\xi, \omega_k(\xi, \eta, \omega_j(x_j)))u_l(\xi, \omega_k(\xi, \eta, \omega_j(x_j)))\, d\xi d\eta$$

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for an arbitrary fixed $j \leq n$. After changing the order of integration we get the equality

$$(RBu)_j(x,t) = -c_j(x_j, x, t) \sum_{k \neq l} \int_0^1 \int_\xi^{1-x_k} r_{jk}(\eta, \omega_j(x_j)) d_k(\xi, \eta, \omega_j(x_j))$$

$$\times b_{kl}(\xi, \omega_k(\xi, \eta, \omega_j(x_j))) u_l(\xi, \omega_k(\xi, \eta, \omega_j(x_j))) \, d\eta d\xi.$$

Then we change the variable η to $z = \omega_k(\xi, \eta, \omega_j(x_j))$. Since the inverse is given by $\eta = \tilde{\omega}_k(\omega_j(x_j), \xi, z)$, we get

$$(RBu)_j(x,t) =$$

$$-c_j(x_j, x, t) \sum_{k \neq l} \int_0^1 \int_{\omega_j(x_j)}^{\omega_k(\xi, 1-x_k, \omega_j(x_j))} r_{jk}(\tilde{\omega}_k(\omega_j(x_j), \xi, z), \omega_j(x_j)) \tag{2.13}$$

$$\times d_k(\xi, \tilde{\omega}_k(\omega_j(x_j), \xi, z), \omega_j(x_j)) b_{kl}(\xi, z) \partial_3 \tilde{\omega}_k(\omega_j(x_j), \xi, z) u_l(\xi, z) \, dz d\xi,$$

where $\partial_3 \tilde{\omega}_k(\omega_j(x_j), \xi, z)$ is given by (2.12). The functions $\omega_j(\xi, x, t)$ and the kernels of the integral operators in (2.13) are continuous and t-periodic functions and, hence, are uniformly continuous functions in x and t. This means that we are again in the conditions of the Arzela-Ascoli theorem, as desired.

We proceed to show that $B^2 : C_{n,2\pi} \to C_{n,2\pi}$ is compact. By the Arcela-Ascoli theorem, $C_{n,2\pi}^1$ is compactly embedded into $C_{n,2\pi}$. Then the desired compactness property will follow if we show that

$$B^2 \text{ maps continuously } C_{n,2\pi} \text{ into } C_{n,2\pi}^1. \tag{2.14}$$

By using the equalities (1.8), (1.9), and (2.1), the partial derivatives $\partial_x B^2 u$, $\partial_t B^2 u$ exist and are continuous for each $u \in C_{n,2\pi}^1$. Since $C_{n,2\pi}^1$ is dense in $C_{n,2\pi}$, the desired condition (2.14) will follow from the bound

$$\left\| B^2 u \right\|_1 = O\left(\|u\|_\infty \right) \text{ for all } u \in C_{n,2\pi}^1. \tag{2.15}$$

To prove (2.15), for given $j \leq n$ and $u \in C_{n,2\pi}^1$, let us consider the following representation for $(B^2 u)_j(x,t)$ obtained after the application of the Fubini's theorem:

$$(B^2 u)_j(x,t) = \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_\eta^x d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \, d\xi d\eta, \tag{2.16}$$

where

$$d_{jkl}(\xi, \eta, x, t) = d_j(\xi, x, t) d_k(\eta, \xi, \omega_j(\xi)) b_{kl}(\eta, \omega_k(\eta, \xi, \omega_j(\xi))). \tag{2.17}$$

The estimate $\|B^2 u\|_\infty = O\left(\|u\|_\infty \right)$ is obvious. Since

$$(\partial_t + a_j(x,t)\partial_x)\varphi(\omega_j(\xi, x, t)) = 0$$

for all $j \leq n, \varphi \in C^1(\mathbb{R}), x, \xi \in [0,1]$, and $t \in \mathbb{R}$, one can easily check that

$$\|[(\partial_t + a_j(x,t)\partial_x)(B^2 u)_j]\|_\infty = O\left(\|u\|_\infty \right) \text{ for all } j \leq n \text{ and } u \in C_{n,2\pi}^1.$$

Hence the estimate $\|\partial_x B^2 u\|_\infty = O\left(\|u\|_\infty\right)$ will follow from the following one:

$$\|\partial_t B^2 u\|_\infty = O\left(\|u\|_\infty\right). \tag{2.18}$$

In order to prove (2.15), we are therefore reduced to prove (2.18). To this end, we start with the following consequence of (2.16):

$$\partial_t[(B^2 u)_j(x,t)]$$
$$= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_\eta^x \frac{d}{dt}\Big[d_{jkl}(\xi,\eta,x,t)b_{jk}(\xi,\omega_j(\xi))\Big]u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi)))\,d\xi d\eta$$
$$+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_\eta^x d_{jkl}(\xi,\eta,x,t)b_{jk}(\xi,\omega_j(\xi))$$
$$\times \partial_t\omega_k(\eta,\xi,\omega_j(\xi))\partial_t\omega_j(\xi)\partial_2 u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi)))\,d\xi d\eta.$$

Let us transform the second summand. Using (1.7), (1.8), and (1.9), we get

$$\frac{d}{d\xi}u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi)))$$
$$= \Big[\partial_x\omega_k(\eta,\xi,\omega_j(\xi)) + \partial_t\omega_k(\eta,\xi,\omega_j(\xi))\partial_\xi\omega_j(\xi)\Big]\partial_2 u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi))) \tag{2.19}$$
$$= \left(\frac{1}{a_j(\xi,\omega_j(\xi))} - \frac{1}{a_k(\xi,\omega_j(\xi))}\right)\partial_t\omega_k(\eta,\xi,\omega_j(\xi))\partial_2 u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi))).$$

Therefore,

$$b_{jk}(\xi,\omega_j(\xi))\partial_t\omega_k(\eta,\xi,\omega_j(\xi))\partial_2 u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi)))$$
$$= a_j(\xi,\omega_j(\xi))a_k(\xi,\omega_j(\xi))\tilde{b}_{jk}(\xi,\omega_j(\xi))\frac{d}{d\xi}u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi))), \tag{2.20}$$

where the functions $\tilde{b}_{jk} \in C_{2\pi}$ are fixed to satisfy (1.6). Note that $\tilde{b}_{jk}$ are not uniquely defined by (1.6) for (x,t) with $a_j(x,t) = a_k(x,t)$. Nevertheless, as it follows from (2.19), the right-hand side (and, hence, the left-hand side of (2.20)) do not depend on the choice of $\tilde{b}_{jk}$, since $\frac{d}{d\xi}u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi))) = 0$ if $a_j(x,t) = a_k(x,t)$.

Write

$$\tilde{d}_{jkl}(\xi,\eta,x,t) = d_{jkl}(\xi,\eta,x,t)\partial_t\omega_j(\xi)a_k(\xi,\omega_j(\xi))a_j(\xi,\omega_j(\xi))\tilde{b}_{jk}(\xi,\omega_j(\xi)),$$

where d_{jkl} are introduced by (2.17) and (1.10). Using (1.7) and (1.8), we see that the function $\tilde{d}_{jkl}(\xi,\eta,x,t)$ is C^1-regular in ξ due to regularity assumptions (1.4) and (1.6). Similarly, using (1.9), we see that the functions $d_{jkl}(\xi,\eta,x,t)$ and $b_{jk}(\xi,\omega_j(\xi))$ are C^1-smooth in t.

By (2.20) we have

$$(\partial_t B^2 u)_j(x,t)$$
$$= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_\eta^x \frac{d}{dt}[d_{jkl}(\xi,\eta,x,t)b_{jk}(\xi,\omega_j(\xi))]u_l(\eta,\omega_k(\eta,\xi,\omega_j(\xi)))\,d\xi d\eta$$

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$$+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^{x} \int_{\eta}^{x} \tilde{d}_{jkl}(\xi, \eta, x, t) \frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \, d\xi d\eta$$

$$= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^{x} \int_{\eta}^{x} \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi))] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \, d\xi d\eta$$

$$- \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^{x} \int_{\eta}^{x} \partial_\xi \tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \, d\xi d\eta$$

$$+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^{x} \left[\tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \right]_{\xi=\eta}^{\xi=x} d\eta.$$

The desired estimate (2.18) now easily follows from the assumptions (1.4)–(1.6).

Returning back to (2.4), it remains to prove that the operator $BR : C_{n,2\pi} \to C_{n,2\pi}$ is compact. By the definitions of B and R,

$$(BRu)_j(x, t) = - \sum_{k \neq j} \sum_{l=1}^{n} \int_0^1 \int_{x_j}^{x} d_j(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) c_k(x_k, \xi, \omega_j(\xi))$$
$$\times r_{kl}(\eta, \omega_k(x_k, \xi, \omega_j(\xi))) u_l(\eta, \omega_k(x_k, \xi, \omega_j(\xi))) \, d\xi d\eta, \quad j \leq n. \tag{2.21}$$

The integral operators in (2.21) are similar to those in (2.16) and, therefore, the proof of the compactness of BR follows along the same line as the proof of the compactness of B^2. The proof of Theorem 1.2 is complete.

Acknowledgments

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